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A new model of relativistic equation of state in accretion and wind flows using 4-velocity distribution function

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Abstract. Following the original line of argument by Maxwell and Boltzmann (MB) we derive a 4-velocity distribution function for a relativistic ideal gas of massive particles. Most importantly, this distribution function can be factorized and perfectly reduces to non-relativistic MB speed distribution formula in low temperature (non relativistic) limit. Using this distribution function we express the pressure p, and kinetic energy density $\rho - \rho_0$ as the functions of a parameter λ directly related to the kinetic energy density and hence to the temperature. We compute the adiabatic index $\gamma = \frac{c_p}{c_v}$ from the relativistic equation of state $\rho - \rho_0 = (\gamma - 1)p$ as a function of the parameter λ. The value of γ exactly reduces to $\frac{5}{3}$ and $\frac{4}{3}$ in the non-relativistic and ultra-relativistic limit respectively. We also find the sound speed(a_s) satisfies $a_s \leq \frac{1}{\sqrt{3}}$ (Basu & Mondal 2013, 2011; Mondal & Basu 2011, 2013).

Keywords : accretion, accretion discs – equation of state – relativity

1. Introduction

The correct choice of the equation of state(EOS) plays a crucial role in the solution of hydrodynamical equations both in the relativistic and non relativistic (NR) case. In the conventional study of accretion and wind flow around a compact star people use the value of adiabatic index $\gamma = \frac{c_p}{c_v}$ as $\frac{4}{3}$ to incorporate ultra relativistic effect (Chakrabarti 1996; Mondal 2010). This approach although counts some of the relativistic effects, misses an essential feature of relativity namely, the dependence of γ with temperature. The direct application of exact relativistic EOS derived using relativistic canonical distribution function (Chandrasekhar 1939; Synge 1957) become less feasible for numerical computing because the distribution function can not be factorized. In our work, we derived a distribution function for 4-velocities (for particles

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with non zero rest mass) following the original approach of Maxwell-Boltzmann used in deriving speed distribution formula for a NR ideal gas (Basu & Mondal 2013, 2011; Mondal & Basu 2011). Our distribution formula can be factorized and therefore, EOS will be convenient to use in many cases (Mondal & Basu 2011, 2013).

2. Four velocity distribution of a relativistic ideal gas

By relativistic gas we mean a gas with very high temperature so that the average thermal energy of a gas particle is comparable to its rest mass energy. For a NR ideal gas in thermal equilibrium at a temperature T, the probability of a gas particle lying between the speed interval v to $v + dv$ is given according to well-known Maxwell-Boltzmann speed distribution formula as

$$P(v)dv = 4\pi A^3 e^{-\frac{m}{2kT}v^2} v^2 dv, \tag{1}$$

where, $A = \sqrt{\frac{m}{2\pi kT}}$ and k is the Boltzmann constant. This formula is derived on the basis of two assumptions: first, the distribution is isotropic and second, the three components of the velocities are statistically independent variable, *i.e.*, any one of the velocity component can take any value between $-\infty$ to ∞ irrespective of the values of the other two components. If $f_i(v^i)dv^i$ is the probability that the particle's i^{th} velocity component will lie in the range v^i to $v^i + dv^i$, then the first assumption enables one to write $f_1 = f_2 = f_3$ (distribution formula is direction independent on account of isotropy). The second assumption that v^is are statistically independent variables, allows one to write the total probability distribution function in a factorized form:

$$P(v^1, v^2, v^3)dv^1dv^2dv^3 = f_1(v^1) f_2(v^2) f_3(v^3)dv^1dv^2dv^3. \tag{2}$$

Here, $P(v^1, v^2, v^3)dv^1dv^2dv^3$ represents the probability that the particle will be simultaneously in the velocity range v_1 to $v_1 + dv_1$, v_2 to $v_2 + dv_2$ and v_3 to $v_3 + dv_3$, which on account of isotropy is again a function independent of the direction of velocity and hence, can be written as $P(v_1, v_2, v_3) = P(v^2)$. Using above equations one can arrive in to MB distribution formula by a straightforward application of calculus (Basu & Mondal 2013, 2011; Mondal & Basu 2011). The second assumption no longer holds for a relativistic gas as the coordinate velocity components $v^i = \frac{dx^i}{dt}$ satisfy the inequality $(v^1)^2 + (v^2)^2 + (v^3)^2 \leq c^2$, where c is the speed of light. Therefore, the range of value that a particular component can take, depends upon the magnitudes of the other two components. However, one can get rid of this difficulty by considering the four velocity components defined as $u^\mu = \frac{1}{m}p^\mu = c\frac{dx^\mu}{d\tau}$ ($x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$), where, p^μ is the four momentum and $d\tau^2 = c^2dt^2 - (d\vec{x})^2$ is the length element in the Minkowski space-time. Four velocity defined above is a physically measurable quantity for only massive particles ($m \neq 0$ and p^μ is a physically measurable quantity) but can not be defined for mass less particles (since $d\tau = 0$, $m = 0$). In this case only four momentum p^μ is defined. Restricting in to the case of massive particles only, one finds that the space-components (u^i)s satisfy the equation

$$\sum_{i=1}^{3}(u^i)^2 = c^2[(u^0)^2 - 1] = v^2/(1 - \frac{v^2}{c^2}). \tag{3}$$

As $v \rightarrow c$ $u^0 \rightarrow \infty$ therefore, each of the u^i can vary form $-\infty$ to ∞ irrespective of the magnitudes of other components. Thus just like the coordinate velocity components in the NR case, the space components of the four velocity are free of constraint and behave as statistically independent variables. In a Lorentz frame, where the centre of mass of the gas container is at rest (or in the co-moving frame of a fluid) the distribution of four velocity is isotropic. Hence for such frame, following the methods of MB in a similar way as in the case of NR gas, we can derive a distribution formula for four velocity as Basu & Mondal (2013, 2011); Mondal & Basu (2011)

$$F(u)du = 4\pi A^3 u^2 e^{-\lambda u^2} du. \tag{4}$$

The normalization condition: $\int_0^\infty F(u)du = 1$ relates A with λ as $A = \sqrt{\frac{\lambda}{\pi}}$. As in the case of MB-distribution the parameter λ in this case is also related to the average kinetic energy ($<E_k>$) as

$$< E_k(\lambda) >= mc^2[< u^0 > -1] = \left[4\pi mc^2 A^3 \int_0^\infty u^2 \sqrt{(1 + \frac{u^2}{c^2})}\, e^{-\lambda u^2} du\right] - mc^2, \tag{5}$$

where m is the rest mass of a gas molecule. The above equation gives the physical meaning of the constant λ and one can compare it with the temperature T in the canonical distribution formula as $< E_k(\lambda) >= mc^2[< u^0 > -1] = \frac{3}{\Theta^2} - \frac{1}{\Theta}\frac{K_1(\Theta)}{K_2(\Theta)} - 1$, where $\Theta = mc^2/kT$ and K_1, K_2 are Bessel's function of second kind. Since $< E_k >$ and T are monotonically decreasing function of λ. Therefore, the NR limit is achieved in the large λ limit. The function $F(u)$ has non-negligible value only in the region where $\lambda u^2 \sim 1$ or, $u^2 \sim \frac{1}{\lambda}$. For large λ, this implies $u^2 = v^2/(1 - \frac{v^2}{c^2}) \ll 1$, i.e. $\frac{v^2}{c^2} \ll 1$. This gives $u^i \approx v^i$, therefore, $F(u)du = 4\pi A^3 v^2 e^{-\lambda v^2} dv$, is the velocity distribution function in the NR case (Basu & Mondal 2013, 2011; Mondal & Basu 2011).

3. The relativistic equation of state

The relativistic EOS is expressed as $\rho - \rho_0 = (\gamma - 1)^{-1} p$. Here, ρ_0 is the rest mass and $\rho = \rho_0 < u^0 >$ is the total energy density. The pressure p, which is the momentum flux per unit area averaged over all the molecules, is given by. $p = \rho_0 < u^i v^i >= \frac{1}{3}\rho_0\Big(< u^0 > - \Big\langle \frac{1}{u^0} \Big\rangle\Big)$ (Basu & Mondal 2011; Mondal & Basu 2011). Using the distribution formula eqn.4 one can calculate p, ρ and γ as the function of λ. The sound speed can be computed as form the equation $a_s^2 = (\frac{\partial p}{\partial \rho})_s$, where s is the specific entropy. The plot of γ and a_s are shown in the left and right panels of Fig. 1 respectively. We notice that they satisfy the correct limiting values $\frac{4}{3} \leq \gamma \leq \frac{5}{3}$ and $a_s \leq \frac{1}{\sqrt{3}}$ and the EOS mostly remains NR except for very low λ (Basu & Mondal 2013, 2011; Mondal & Basu 2011).

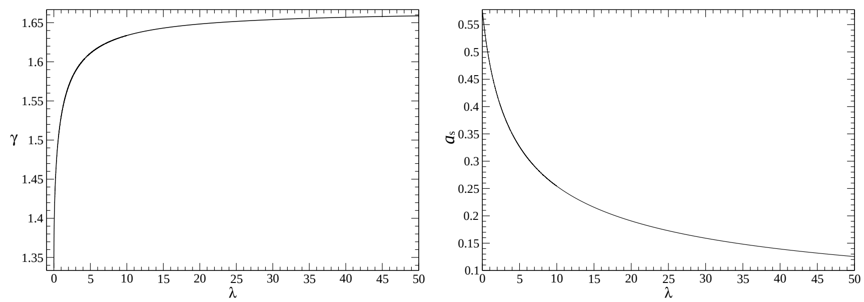

Figure 1. *Left panel:* Variation of γ with λ. *Right Panel:* Variation of sound speed a_s with λ.

This equation of state works only for massive particles. For massless particles γ can be calculated using canonical distribution formula and one finds that $\gamma = \frac{4}{3}$ and independent of temperature.

4. Conclusion

Using assumption of isotropy and statistical independence of the 4-velocity components we derived a distribution function of the 4-velocity for a relativistic ideal gas of massive particles. Our approach is an extension of the Maxwell-Boltzmann arguments used in deriving the speed distribution formula for non relativistic (NR) ideal gas. We then computed γ and a_s (Basu & Mondal 2013, 2011; Mondal & Basu 2011). The plots of these quantities show that they match perfectly well to their limiting values for both NR and extreme relativistic region. It is also observed that the EOS mostly remains NR except for very high value of λ. An application of these EOS in accretion and wind flows has been discussed elsewhere (Mondal & Basu 2011, 2013).

Editor's Comment: *A difference of opinion between authors and referees is observed in the context of distribution function in the four velocity space.*

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